

# Maps, Hash Tables and Dictionaries

## Chapter 9



# Outline

- Maps
- Hashing
- Dictionaries
- Ordered Maps & Dictionaries
  - Iterative Algorithms: Case Study of Binary Search
    - ✧ Loop Invariants
    - ✧ Boundary Conditions

# Maps



- A map models a searchable collection of key-value entries
- The main operations of a map are for searching, inserting, and deleting items
- Multiple entries with the same key are **not** allowed
- Applications:
  - ❑ address book
  - ❑ student-record database

# The Map ADT (`net.datastructures.Map`)

## ➤ Map ADT methods:

- ❑ `get(k)`: if the map `M` has an entry with key `k`, return its associated value; else, return `null`
- ❑ `put(k, v)`: insert entry `(k, v)` into the map `M`; if key `k` is not already in `M`, then return `null`; else, return old value associated with `k`
- ❑ `remove(k)`: if the map `M` has an entry with key `k`, remove it from `M` and return its associated value; else, return `null`
- ❑ `size()`, `isEmpty()`
- ❑ `keys()`: return an iterator over the keys in `M`
- ❑ `values()`: return an iterator of the values in `M`
- ❑ `entries()`: returns an iterator over the entries in `M`

# Example

<b>Operation</b>	<b>Output</b>	<b>M</b>
isEmpty()	true	$\emptyset$
put(5,A)	null	(5,A)
put(7,B)	null	(5,A),(7,B)
put(2,C)	null	(5,A),(7,B),(2,C)
put(8,D)	null	(5,A),(7,B),(2,C),(8,D)
put(2,E)	C	(5,A),(7,B),(2,E),(8,D)
get(7)	B	(5,A),(7,B),(2,E),(8,D)
get(4)	null	(5,A),(7,B),(2,E),(8,D)
get(2)	E	(5,A),(7,B),(2,E),(8,D)
size()	4	(5,A),(7,B),(2,E),(8,D)
remove(5)	A	(7,B),(2,E),(8,D)
remove(2)	E	(7,B),(8,D)
get(2)	null	(7,B),(8,D)
isEmpty()	false	(7,B),(8,D)

# Comparison with java.util.Map

## Map ADT Methods

size()

isEmpty()

get(*k*)

put(*k*, *v*)

remove(*k*)

keys()

values()

entries()

## java.util.Map Methods

size()

isEmpty()

get(*k*)

put(*k*, *v*)

remove(*k*)

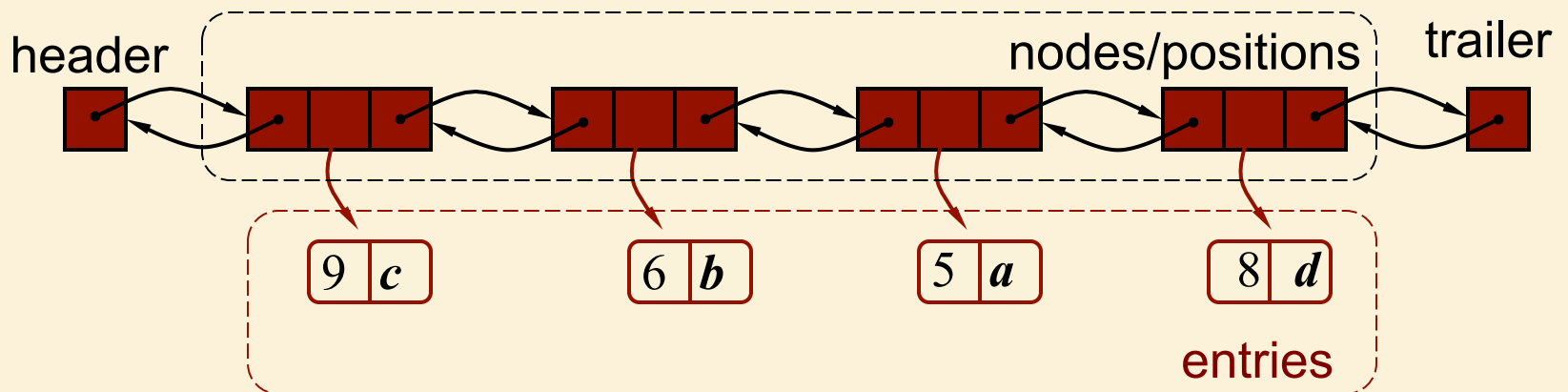
keySet()

values()

entrySet()

# A Simple List-Based Map

- We could implement a map using an unsorted list
  - We store the entries of the map in a doubly-linked list *S*, in arbitrary order
  - *S* supports the node list ADT (Section 6.2)



# The get(k) Algorithm

**Algorithm** get( $k$ ):

$B = S.\text{positions}()$  *{ $B$  is an iterator of the positions in  $S$ }*

**while**  $B.\text{hasNext}()$  **do**

$p = B.\text{next}()$  // the next position in  $B$

**if**  $p.\text{element}().\text{getKey}() = k$  **then**

**return**  $p.\text{element}().\text{getValue}()$

**return null** *{there is no entry with key equal to  $k$ }*



# The put(*k*,*v*) Algorithm

**Algorithm** put(*k*,*v*):

*B* = *S*.positions()

**while** *B*.hasNext() **do**

*p* = *B*.next()

**if** *p*.element().getKey() = *k* **then**

*t* = *p*.element().getValue()

*S*.set(*p*,(*k*,*v*))

**return** *t* {return the old value}

*S*.addLast((*k*,*v*))

*n* = *n* + 1                    {increment variable storing number of entries}

**return** null                    {there was no previous entry with key equal to *k*}

# The remove(*k*) Algorithm

**Algorithm** remove(*k*):

*B* = *S*.positions()

**while** *B*.hasNext() **do**

*p* = *B*.next()

**if** *p*.element().getKey() = *k* **then**

*t* = *p*.element().getValue()

*S*.remove(*p*)

*n* = *n* − 1      {decrement number of entries}

**return** *t*      {return the removed value}

**return null**      {there is no entry with key equal to *k*}

# Performance of a List-Based Map

## ➤ Performance:

❑ **put**, **get** and **remove** take  $O(n)$  time since in the worst case (the item is not found) we traverse the entire sequence to look for an item with the given key

➤ The unsorted list implementation is effective only for small maps

# Hash Tables

- A hash table is a data structure that can be used to make map operations faster.
- While worst-case is still  $O(n)$ , average case is typically  $O(1)$ .

# Applications of Hash Tables

- databases
- compilers
- browser caches

# Hash Functions and Hash Tables

- A **hash function**  $h$  maps keys of a given type to integers in a fixed interval  $[0, N-1]$

- Example:

$$h(x) = x \bmod N$$

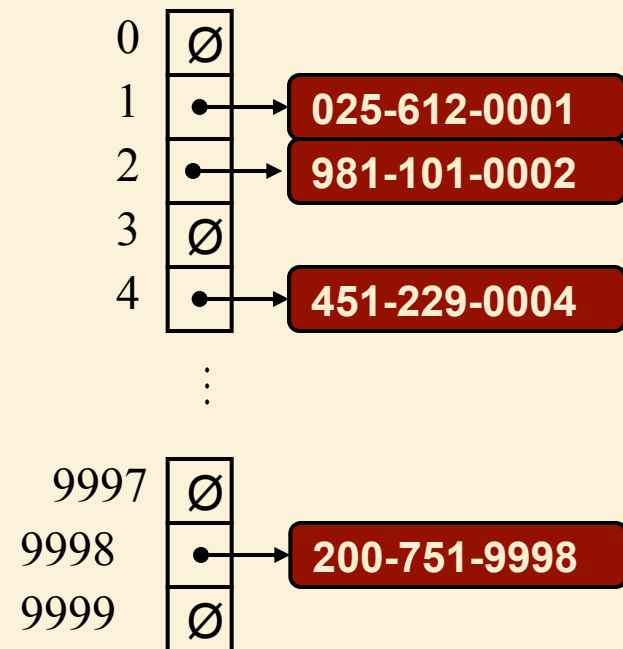
is a hash function for integer keys



- The integer  $h(x)$  is called the **hash value** of key  $x$
- A **hash table** for a given key type consists of
  - ❑ Hash function  $h$
  - ❑ Array (called table) of size  $N$
- When implementing a map with a hash table, the goal is to store item  $(k, o)$  at index  $i = h(k)$

# Example

- We design a hash table for a map storing entries as (SIN, Name), where SIN (social insurance number) is a nine-digit positive integer
- Our hash table uses an array of size  $N=10,000$  and the hash function  $h(x)$  = last four digits of SIN  $x$



# Hash Functions



- A hash function is usually specified as the composition of two functions:

**Hash code:**

$h_1$ : keys  $\rightarrow$  integers

**Compression function:**

$h_2$ : integers  $\rightarrow [0, N-1]$

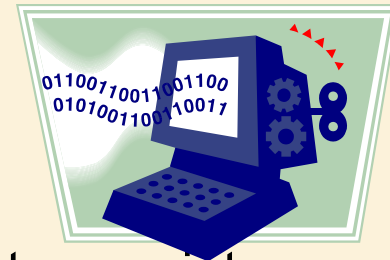
- The hash code is applied first, and the compression function is applied next on the result, i.e.,

$$h(x) = h_2(h_1(x))$$

- The goal of the hash function is to “disperse” the keys in an apparently random way



# Hash Codes



## ➤ Memory address:

- ☐ We reinterpret the memory address of the key object as an integer (default hash code of all Java objects)
- ☐ Does not work well when copies of the same object may be stored at different locations.

## ➤ Integer cast:

- ☐ We reinterpret the bits of the key as an integer
- ☐ Suitable for keys of length less than or equal to the number of bits of the integer type (e.g., byte, short, int and float in Java)

## ➤ Component sum:

- ☐ We partition the bits of the key into components of fixed length (e.g., 16 or 32 bits) and we sum the components (ignoring overflows)
- ☐ Suitable for numeric keys of fixed length greater than or equal to the number of bits of the integer type (e.g., long and double in Java)

# Problems with Component Sum Hash Codes

## ➤ Hashing works when

- ❑ the number of different common keys is small relative to the hashing space (e.g.,  $2^{32}$  for a 32-bit hash code).
- ❑ the hash codes for common keys are well-distributed (do not collide) in this space.

## ➤ Component Sum codes ignore the ordering of the components.

- ❑ e.g., using 8-bit ASCII components, 'stop' and 'pots' yields the same code.

## ➤ Since common keys are often anagrams of each other, this is often a bad idea!

# Polynomial Hash Codes

## ➤ Polynomial accumulation:

- We partition the bits of the key into a sequence of components of fixed length (e.g., 8, 16 or 32 bits)

$$a_0 a_1 \dots a_{n-1}$$

- We evaluate the polynomial

$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_{n-1} z^{n-1}$  at a fixed value  $z$ , ignoring overflows

- Especially suitable for strings (e.g., the choice  $z = 33$  gives at most 6 collisions on a set of 50,000 English words)

- Polynomial  $p(z)$  can be evaluated in  $O(n)$  time using Horner's rule:

✧ The following polynomials are successively computed, each from the previous one in  $O(1)$  time

$$p_0(z) = a_{n-1}$$

$$p_i(z) = a_{n-i-1} + z p_{i-1}(z) \quad (i = 1, 2, \dots, n-1)$$

- We have  $p(z) = p_{n-1}(z)$

# Compression Functions

## ➤ Division:

- ❑  $h_2(y) = y \bmod N$

- ❑ The size  $N$  of the hash table is usually chosen to be a prime (on the assumption that the differences between hash keys  $y$  are less likely to be multiples of primes).

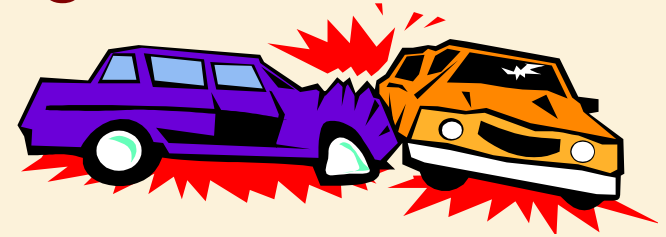
## ➤ Multiply, Add and Divide (MAD):

- ❑  $h_2(y) = [(ay + b) \bmod p] \bmod N$ , where

- ✧  $p$  is a prime number greater than  $N$

- ✧  $a$  and  $b$  are integers chosen at random from the interval  $[0, p - 1]$ , with  $a > 0$ .

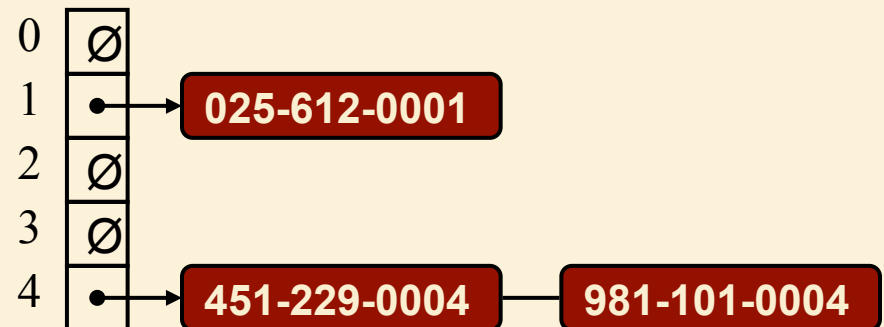
# Collision Handling



- Collisions occur when different elements are mapped to the same cell

- **Separate Chaining:**

- ❑ Let each cell in the table point to a linked list of entries that map there
- ❑ Separate chaining is simple, but requires additional memory outside the table



# Map Methods with Separate Chaining

➤ Delegate operations to a list-based map at each cell:

**Algorithm**  $\text{get}(k)$ :

**Output:** The value associated with the key  $k$  in the map, or **null** if there is no entry with key equal to  $k$  in the map

**return**  $A[h(k)].\text{get}(k)$       {delegate the get to the list-based map at  $A[h(k)]$ }

# Map Methods with Separate Chaining

➤ Delegate operations to a list-based map at each cell:

**Algorithm**  $\text{put}(k, v)$ :

**Output:** Store the new (key, value) pair. If there is an existing entry with key equal to  $k$ , return the old value; otherwise, return **null**

$t = A[h(k)].\text{put}(k, v)$       {delegate the put to the list-based map at  $A[h(k)]$ }

**if**  $t = \text{null}$  **then**      { $k$  is a new key}

$n = n + 1$

**return**  $t$

# Map Methods with Separate Chaining

➤ Delegate operations to a list-based map at each cell:

**Algorithm** `remove(k)`:

**Output:** The (removed) value associated with key  $k$  in the map, or **null** if there is no entry with key equal to  $k$  in the map

$t = A[h(k)].remove(k)$       {delegate the remove to the list-based map at  $A[h(k)]$ }

**if**  $t \neq \text{null}$  **then**      { $k$  was found}

$n = n - 1$

**return**  $t$



# Linear Probing

- **Open addressing**: the colliding item is placed in a different cell of the table
- **Linear probing** handles collisions by placing the colliding item in the next (circularly) available table cell
- Each table cell inspected is referred to as a “probe”
- Colliding items lump together, so that future collisions cause a longer sequence of probes

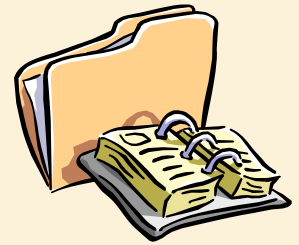
- Example:

□  $h(x) = x \bmod 13$

- Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

		41			18	44	59	32	22	31	73	
0	1	2	3	4	5	6	7	8	9	10	11	12

# Get with Linear Probing



➤ Consider a hash table **A** of length **N** that uses linear probing

➤ **get(k)**

- ❑ We start at cell  $h(k)$
- ❑ We probe consecutive locations until one of the following occurs
  - ✧ An item with key  $k$  is found, or
  - ✧ An empty cell is found, or
  - ✧  $N$  cells have been unsuccessfully probed

**Algorithm** *get(k)*

$i \leftarrow h(k)$

$p \leftarrow 0$

repeat

$c \leftarrow A[i]$

if  $c = \emptyset$

return *null*

else if  $c.key() = k$

return  $c.element()$

else

$i \leftarrow (i + 1) \bmod N$

$p \leftarrow p + 1$

until  $p = N$

return *null*

# Remove with Linear Probing

➤ Suppose we receive a **remove(44)** message.

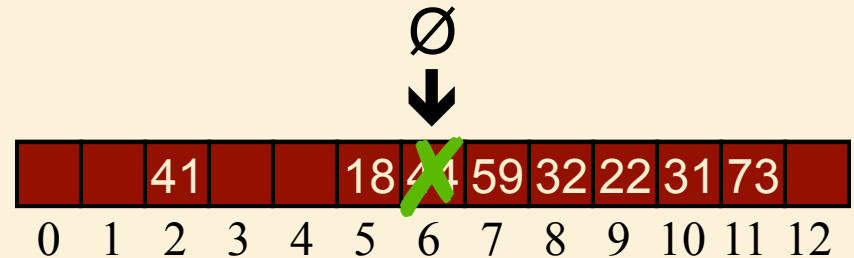
➤ What problem arises if we simply remove the key = 44 entry?

➤ Example:

□  $h(x) = x \bmod 13$

□ Insert keys 18, 41, 22, 44, 59, 32, 31, 73, in this order

<i>k</i>	<i>h(k)</i>	<i>i</i>
18	5	5
41	2	2
22	9	9
44	5	6
59	7	7
32	6	8
31	5	10
73	8	11



# Removal with Linear Probing

- To address this problem, we introduce a special object, called **AVAILABLE**, which replaces deleted elements
- **AVAILABLE** has a null key
- No changes to **get**( $k$ ) are required.

**Algorithm** *get*( $k$ )

$i \leftarrow h(k)$

$p \leftarrow 0$

repeat

$c \leftarrow A[i]$

if  $c = \emptyset$

return *null*

else if  $c.key() = k$

return  $c.element()$

else

$i \leftarrow (i + 1) \bmod N$

$p \leftarrow p + 1$

until  $p = N$

return *null*

# Updates with Linear Probing

## ➤ **remove( $k$ )**

- ❑ We search for an entry with key  $k$
- ❑ If such an entry  $(k, o)$  is found, we replace it with the special item *AVAILABLE* and we return element  $o$
- ❑ Else, we return *null*

## ➤ **put( $k, o$ )**

- ❑ We throw an exception if the table is full
- ❑ We start at cell  $h(k)$
- ❑ We probe consecutive cells until one of the following occurs
  - ✧ A cell  $i$  is found that is either empty or stores *AVAILABLE*, or
  - ✧  $N$  cells have been unsuccessfully probed
- ❑ We store entry  $(k, o)$  in cell  $i$

# Double Hashing

- Double hashing uses a **secondary hash function  $h'(k)$**  in addition to the primary hash function  $h(x)$ .
- Suppose that the primary hashing  $i=h(k)$  leads to a collision.
- We then iteratively probe the locations  
 $(i + jh'(k)) \bmod N$  for  $j = 0, 1, \dots, N - 1$
- The secondary hash function  **$h'(k)$**  cannot have zero values
- **$N$**  is typically chosen to be prime.
- Common choice of secondary hash function  $h'(k)$ :
  - ❑  $h'(k) = q - k \bmod q$ , where
    - ✧  $q < N$
    - ✧  $q$  is a prime
- The possible values for  $h'(k)$  are  
 $1, 2, \dots, q$

# Example of Double Hashing

- Consider a hash table storing integer keys that handles collision with double hashing

□  $N=13$

□  $h(k) = k \bmod 13$

□  $h'(k) = 7 - k \bmod 7$

- Insert keys 18, 41, 22, 44, 59, 32, 31, 73

$k$	$h(k)$	$h'(k)$	Probes	
18	5	3	5	
41	2	1	2	
22	9	6	9	
44	5	5	5	10
59	7	4	7	
32	6	3	6	
31	5	4	5	9 0
73	8	4	8	

31		41			18	32	59	73	22	44		
0	1	2	3	4	5	6	7	8	9	10	11	12

# End of Lecture

Feb 7, 2012



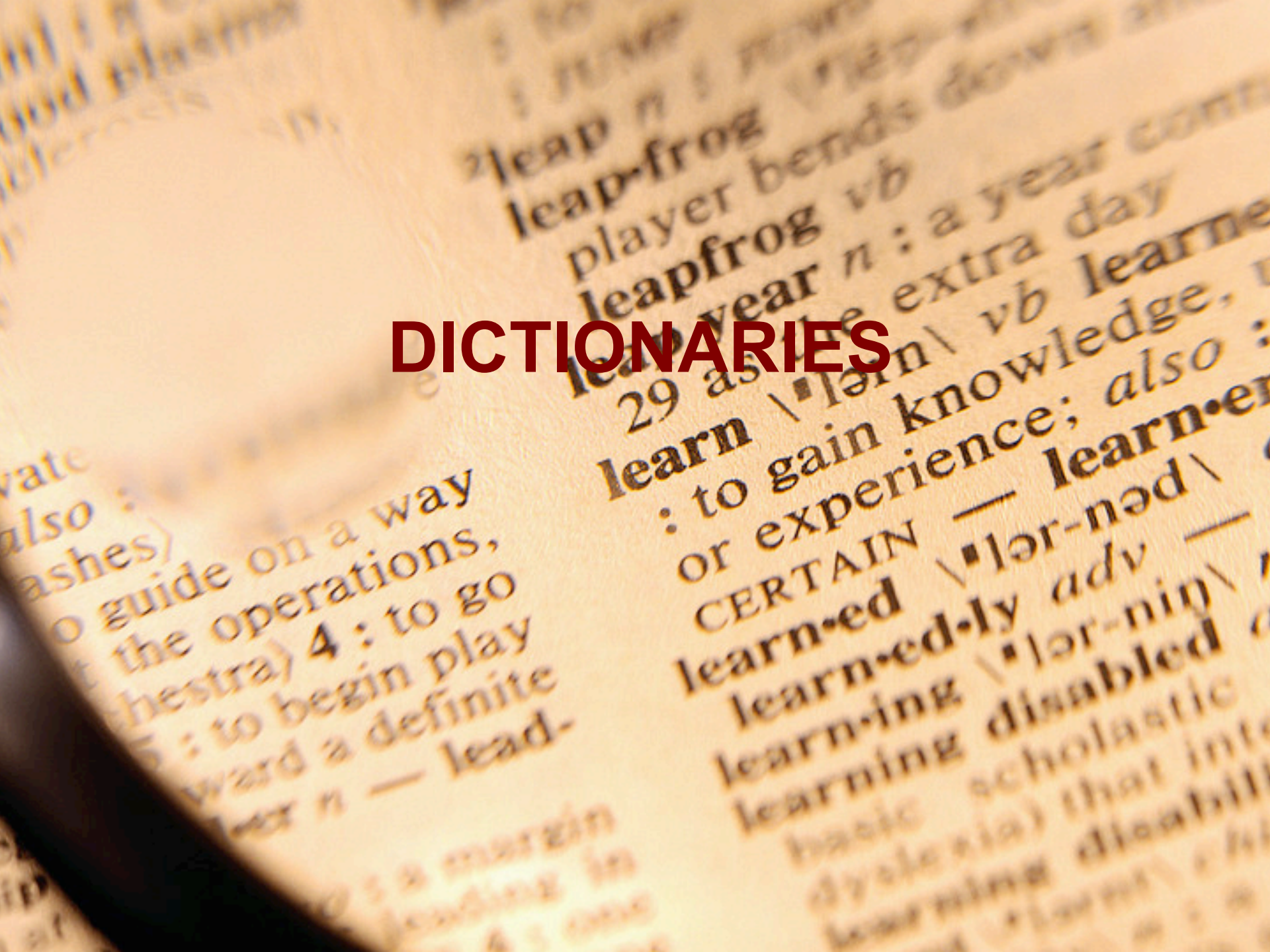
# Performance of Hashing

- In the worst case, searches, insertions and removals on a hash table take  $O(n)$  time
- The worst case occurs when all the keys inserted into the map collide
- The load factor  $\lambda = n/N$  affects the performance of a hash table
  - ❑ For separate chaining, performance is typically good for  $\lambda < 0.9$ .
  - ❑ For open addressing , performance is typically good for  $\lambda < 0.5$ .
  - ❑ `java.util.HashMap` maintains  $\lambda < 0.75$
- Separate chaining is typically as fast or faster than open addressing.

# Rehashing

- When the load factor  $\lambda$  exceeds threshold, the table must be **rehashed**.
  - ❑ A larger table is allocated (typically at least double the size).
  - ❑ A new hash function is defined.
  - ❑ All existing entries are copied to this new table using the new hash function.

# DICTIONARIES



# Dictionary ADT

- The dictionary ADT models a searchable collection of key-element entries
- The main operations of a dictionary are searching, inserting, and deleting items
- Multiple items with the same key **are** allowed
- Applications:
  - ❑ word-definition pairs
  - ❑ credit card authorizations
  - ❑ DNS mapping of host names (e.g., datastructures.net) to internet IP addresses (e.g., 128.148.34.101)
- Dictionary ADT methods:
  - ❑ **get**(k): if the dictionary has at least one entry with key k, returns one of them, else, returns null
  - ❑ **getAll**(k): returns an iterable collection of all entries with key k
  - ❑ **put**(k, v): inserts and returns the entry (k, v)
  - ❑ **remove**(e): removes and returns the entry e. Throws an exception if the entry is not in the dictionary.
  - ❑ **entrySet**(): returns an iterable collection of the entries in the dictionary
  - ❑ **size**(), **isEmpty**()

# The Dictionary ADT in net.datastructures is slightly different

## ➤ Textbook:

- ❑ **get**(k): if the dictionary has at least one entry with key k, returns one of them, else returns null
- ❑ **getAll**(k): returns an iterable collection of all entries with key k
- ❑ **put**(k, v): inserts and returns the entry (k, v)
- ❑ **remove**(e): removes and returns the entry e. Throws an exception if the entry is not in the dictionary.
- ❑ **entrySet**(): returns an iterable collection of the entries in the dictionary
- ❑ **size**(), **isEmpty**()

## ➤ net.datastructures:

- ❑ **find**(k): if the dictionary has at least one entry with key k, returns one of them else, returns null
- ❑ **findAll**(k): returns an iterator of all entries with key k
- ❑ **insert**(k, o): inserts and returns the entry (k, o)
- ❑ **remove**(e): remove the entry e from the dictionary
- ❑ **entries**(): returns an iterator of the entries in the dictionary
- ❑ **size**(), **isEmpty**()



# Dictionaries and Java

- Note: The `java.util.Dictionary` class actually implements a map ADT.
- There is no dictionary data structure in the Java Collections Framework that supports multiple entries with equal keys.
- The textbook (Section 9.5.3) provides an implementation of a Dictionary based upon a map of keys, each entry of which supports a linked list of entries with the same key.

# Example

<b>Operation</b>	<b>Output</b>	<b>Dictionary</b>
put(5,A)	(5,A)	(5,A)
put(7,B)	(7,B)	(5,A),(7,B)
put(2,C)	(2,C)	(5,A),(7,B),(2,C)
put(8,D)	(8,D)	(5,A),(7,B),(2,C),(8,D)
put(2,E)	(2,E)	(5,A),(7,B),(2,C),(8,D),(2,E)
get(7)	(7,B)	(5,A),(7,B),(2,C),(8,D),(2,E)
get(4)	<b>null</b>	(5,A),(7,B),(2,C),(8,D),(2,E)
get(2)	(2,C)	(5,A),(7,B),(2,C),(8,D),(2,E)
getAll(2)	(2,C),(2,E)	(5,A),(7,B),(2,C),(8,D),(2,E)
size()	5	(5,A),(7,B),(2,C),(8,D),(2,E)
remove(get(5))	(5,A)	(7,B),(2,C),(8,D),(2,E)
get(5)	<b>null</b>	(7,B),(2,C),(8,D),(2,E)

# Subtleties of remove(e)

- **remove(e)** will remove an entry that matches **e** (i.e., has the same (key, value) pair).
- If the dictionary contains more than one entry with identical (key, value) pairs, **remove(e)** will only remove one.
- Example:

<i>Operation</i>	<i>Output</i>	<i>Dictionary</i>
e1 = put(2,C)	(2,C)	(5,A),(7,B),(2,C)
e2 = put(8,D)	(8,D)	(5,A),(7,B),(2,C),(8,D)
e3 = put(2,E)	(2,E)	(5,A),(7,B),(2,C),(8,D),(2,E)
remove(get(5))	(5,A)	(7,B),(2,C),(8,D),(2,E)
remove(e3)	(2,E)	(7,B),(2,C),(8,D)
remove(e1)	(2,C)	(7,B),(8,D)



# A List-Based Dictionary

- A log file or audit trail is a dictionary implemented by means of an unsorted sequence
  - ❑ We store the items of the dictionary in a sequence (based on a doubly-linked list or array), in arbitrary order
- Performance:
  - ❑ **insert** takes  $O(1)$  time since we can insert the new item at the beginning or at the end of the sequence
  - ❑ **find** and **remove** take  $O(n)$  time since in the worst case (the item is not found) we traverse the entire sequence to look for an item with the given key
- The log file is effective only for dictionaries of small size or for dictionaries on which insertions are the most common operations, while searches and removals are rarely performed (e.g., historical record of logins to a workstation)

# Hash Table Implementation

- We can also create a hash-table dictionary implementation.
- If we use separate chaining to handle collisions, then each operation can be delegated to a list-based dictionary stored at each hash table cell.

# The getAll and put Algorithms

## Algorithm **getAll**(k)

Create an initially-empty list L

**for** e: D **do**

**if** e.getKey() = k **then**

        L.addLast(e)

**return** L

## Algorithm **put**(k,v)

Create a new entry e = (k,v)

S.addLast(e)     {S is unordered}

**return** e

# The remove Algorithm

**Algorithm** `remove(e)`:

{ We don't assume here that `e` stores its position in `S` }

`B = S.positions()`

**while** `B.hasNext()` **do**

`p = B.next()`

**if** `p.element() = e` **then**

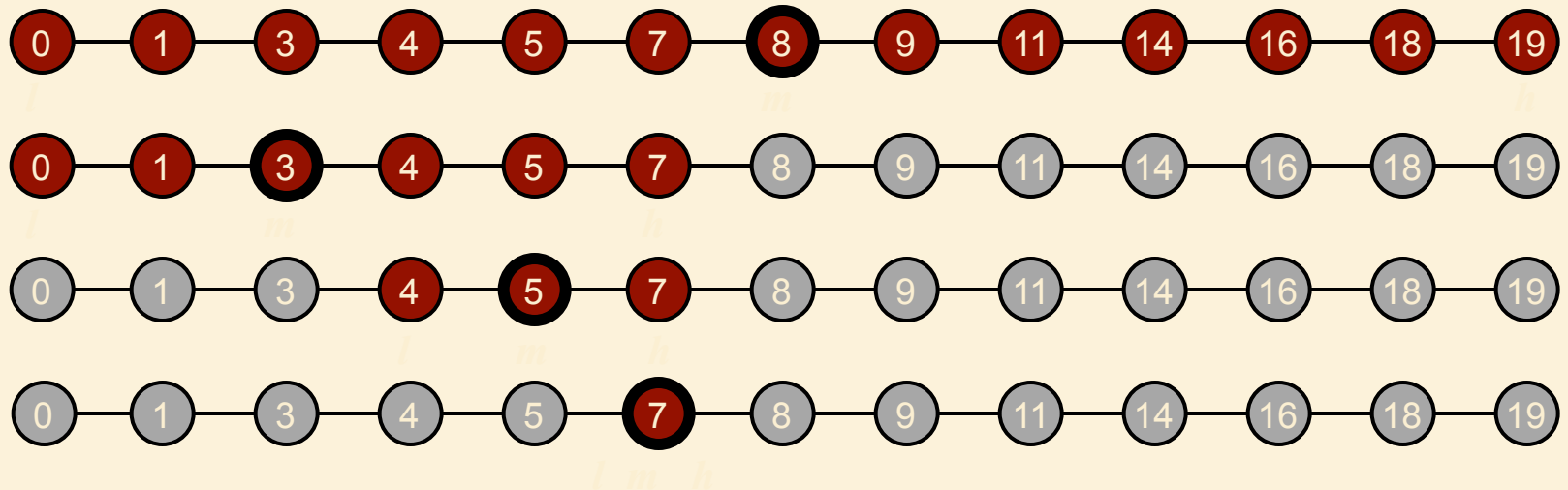
`S.remove(p)`

**return** `e`

**return null** {there is no entry `e` in `D`}

# Ordered Maps and Dictionaries

- If keys obey a total order relation, can represent a map or dictionary as an ordered search table stored in an array.
- Can then support a fast **find(k)** using **binary search**.
  - ❑ at each step, the number of candidate items is halved
  - ❑ terminates after a logarithmic number of steps
  - ❑ Example: **find(7)**



# Ordered Search Tables

## ➤ Performance:

- ❑ **find** takes  $O(\log n)$  time, using binary search

- ❑ **insert** takes  $O(n)$  time since in the worst case we have to shift  $n$  items to make room for the new item

- ❑ **remove** takes  $O(n)$  time since in the worst case we have to shift  $n$  items to compact the items after the removal

➤ A search table is effective only for dictionaries of small size or for dictionaries on which searches are the most common operations, while insertions and removals are rarely performed (e.g., credit card authorizations)

# Binary Search: A Case Study of an Iterative Algorithm

- The algorithm for binary search that we just described is an example of an **iterative algorithm** (note that we could also implement binary search recursively).
- **Loop Invariant:** An **assertion** about the current program state useful for designing, analyzing and proving the correctness of iterative algorithms.
- What is an appropriate loop invariant for our binary search algorithm?
  - ❑ Maintain a sublist.
  - ❑ If the key is contained in the original list, then the key is contained in the sublist.

# Definition of Assertions

An assertion is not a task for the algorithm to perform.

It is only a comment that is added for the benefit of the reader.



# Specifying a Computational Problem

## Other Examples of Assertions

- **Preconditions:** Any assumptions that must be true about the input instance.
- **Postconditions:** The statement of what must be true when the algorithm/program returns.

# Iterative Algorithms

Take one step at a time  
towards the final destination

loop (done)

take step

end loop

# Establishing Loop Invariant

From the Pre-Conditions on the input instance we must establish the loop invariant.



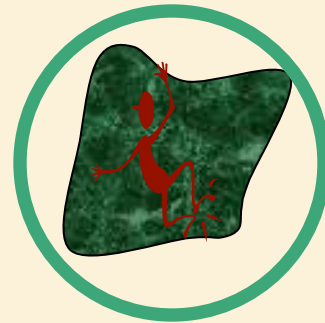
# Maintain Loop Invariant

➤ Suppose that

- ❑ We start in a safe location (pre-condition)
- ❑ If we are in a safe location, we always step to another safe location (loop invariant)

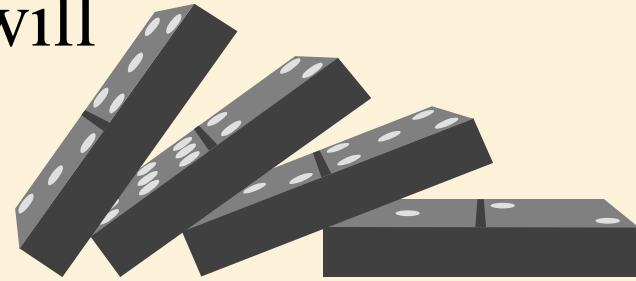
➤ Can we be assured that the computation will always be in a safe location?

➤ By what principle?



# Maintain Loop Invariant

- By Induction the computation will always be in a safe location.

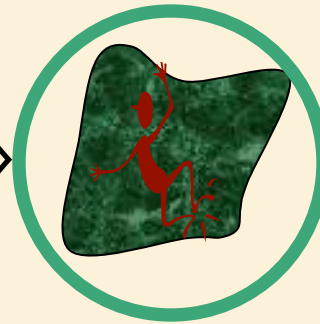


$\Rightarrow S(0)$



$\Rightarrow \forall i, S(i) \Rightarrow S(i+1)$

$\Rightarrow \forall i, S(i) \Rightarrow$

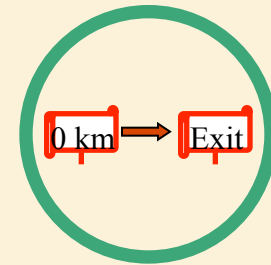


# Ending The Algorithm

- Define Exit Condition



- Termination: With sufficient progress, the exit condition will be met.



- When we exit, we know
  - ❑ exit condition is true
  - ❑ loop invariant is truefrom these we must establish the post conditions.



# Definition of Correctness

$\langle \text{PreCond} \rangle \ \& \ \langle \text{code} \rangle \rightarrow \langle \text{PostCond} \rangle$

If the input meets the preconditions,  
then the output must meet the postconditions.

If the input does not meet the preconditions, then  
nothing is required.

# Define Problem: Binary Search

## ➤ PreConditions

☐ Key     25

☐ Sorted List

3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95
---	---	---	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----

## ➤ PostConditions

☐ Find key in list (if there).

3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95
---	---	---	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----



# Define Loop Invariant

- Maintain a sublist.
- If the key is contained in the original list, then the key is contained in the sublist.

key 25

3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95
---	---	---	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----

# Define Step

- Cut sublist in half.
- Determine which half the key would be in.
- Keep that half.

key 25

mid

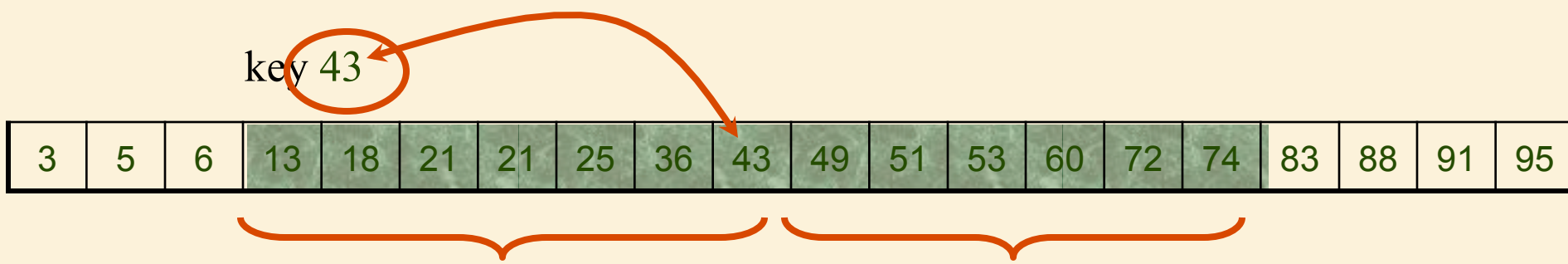
If  $\text{key} \leq \text{mid}$ ,  
then key is in  
left half.

If  $\text{key} > \text{mid}$ ,  
then key is in  
right half.

# Define Step

- It is faster not to check if the middle element is the key.
- Simply continue.

key 43



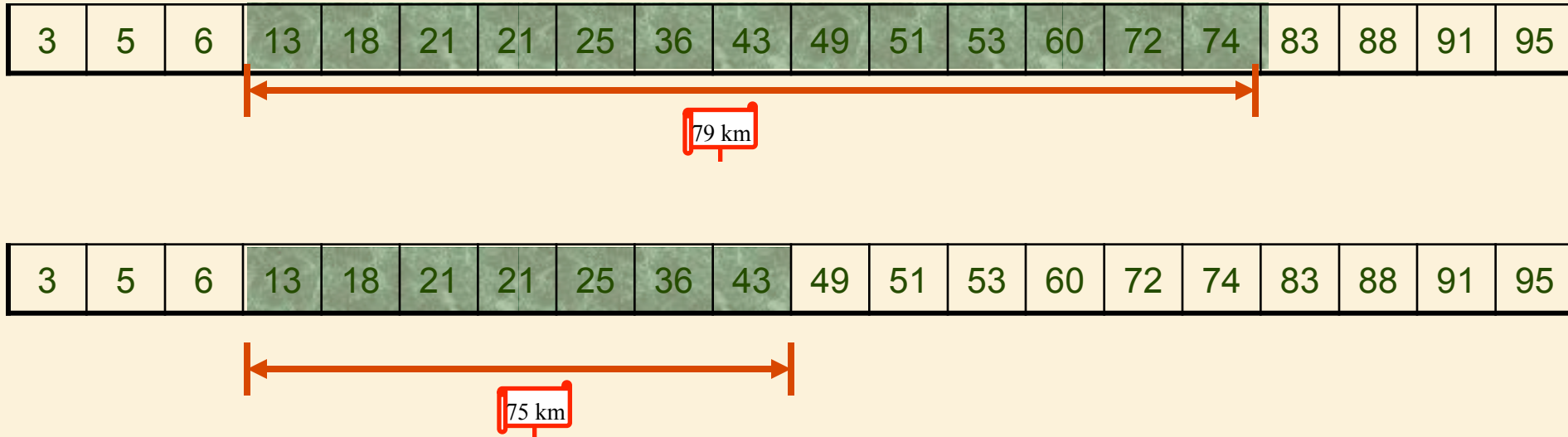
3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95
---	---	---	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----

If  $\text{key} \leq \text{mid}$ ,  
then key is in  
left half.

If  $\text{key} > \text{mid}$ ,  
then key is in  
right half.

# Make Progress

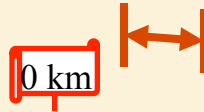
➤ The size of the list becomes smaller.



# Exit Condition

key 25

3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95
---	---	---	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----



- If the key is contained in the original list,  
then the key is contained in the sublist.
- Sublist contains one element.



- If the key is contained in the original list,  
then the key is at this location.

# If key not in original list

- If the key is contained in the original list,  
then the key is contained in the sublist.

- Loop invariant true, even if the key is not in the list.

key 24

3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95
---	---	---	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----

- If the key is contained in the original list, then the key is at this location.
- Conclusion still solves the problem.  
Simply check this one location for the key.

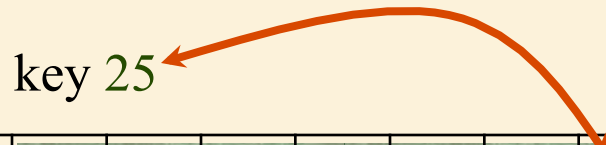
# Running Time

The sublist is of size  $n$ ,  $n/2$ ,  $n/4$ ,  $n/8, \dots, 1$

Each step  $O(1)$  time.

Total =  $O(\log n)$

key 25



If  $\text{key} \leq \text{mid}$ ,  
then key is in  
left half.

If  $\text{key} > \text{mid}$ ,  
then key is in  
right half.

# Running Time

- Binary search can interact poorly with the memory hierarchy (i.e. caching), because of its random-access nature.
- It is common to abandon binary searching for linear searching as soon as the size of the remaining span falls below a small value such as 8 or 16 or even more in recent computers.



BinarySearch( $A[1..n], key$ )

<precondition>:  $A[1..n]$  is sorted in non-decreasing order

<postcondition>: If  $key$  is in  $A[1..n]$ , algorithm returns its location

$p = 1, q = n$

while  $q > p$

<loop-invariant>: If  $key$  is in  $A[1..n]$ , then  $key$  is in  $A[p..q]$

$mid = \left\lfloor \frac{p+q}{2} \right\rfloor$

if  $key \leq A[mid]$

$q = mid$

else

$p = mid + 1$

end

end

if  $key = A[p]$

return( $p$ )

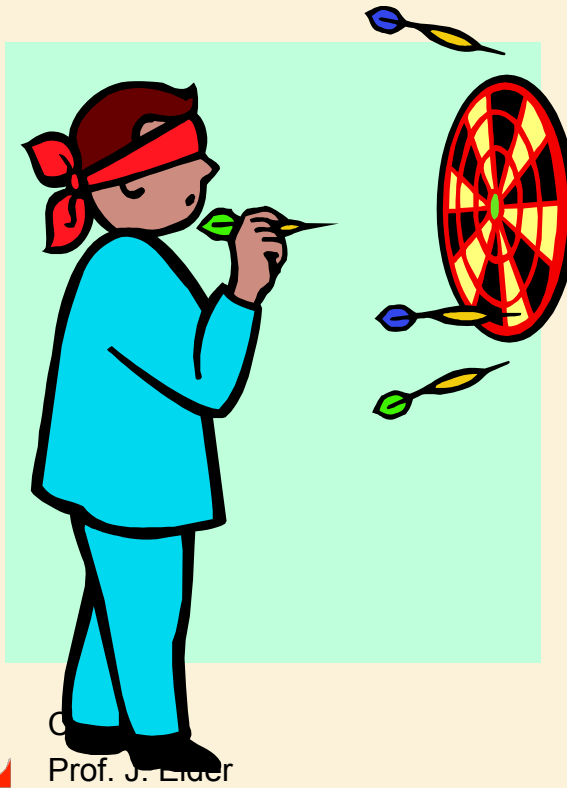
else

return("Key not in list")

end

# Simple, right?

- Although the concept is simple, binary search is notoriously easy to get wrong.
- Why is this?



# End of Lecture

Feb 9, 2012

# Boundary Conditions

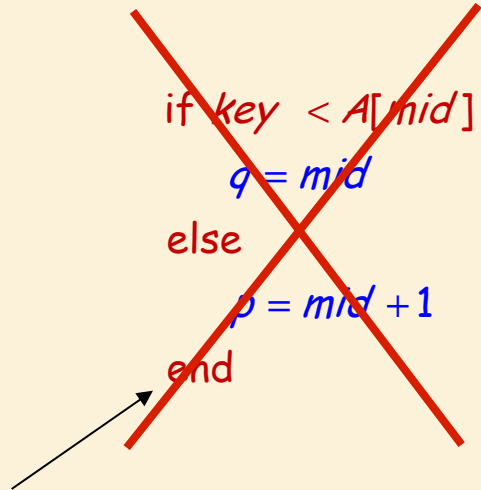
- The basic idea behind binary search is easy to grasp.
- It is then easy to write pseudocode that works for a 'typical' case.
- Unfortunately, it is equally easy to write pseudocode that fails on the *boundary conditions*.

# Boundary Conditions

```
if key ≤ A[mid]  
  q = mid  
else  
  p = mid + 1  
end
```

or

```
if key < A[mid]  
  q = mid  
else  
  p = mid + 1  
end
```



What condition will break the loop invariant?

# Boundary Conditions

key 36

mid



3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95
---	---	---	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----

Code:  $\text{key} \geq A[\text{mid}] \rightarrow$  select right half

Bug!!

# Boundary Conditions

```
if key ≤ A[mid]
  q = mid
else
  p = mid + 1
end
```

OK

```
if key < A[mid]
  q = mid - 1
else
  p = mid
end
```

OK

```
if key < A[mid]
  q = mid
else
  p = mid + 1
end
```

Not OK!!

# Boundary Conditions

$$\text{mid} = \left\lfloor \frac{p+q}{2} \right\rfloor$$

or

$$\text{mid} = \left\lceil \frac{p+q}{2} \right\rceil$$

key 25



3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95
---	---	---	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----

Shouldn't matter, right?

Select  $\text{mid} = \left\lceil \frac{p+q}{2} \right\rceil$



# Boundary Conditions

$$\text{Select mid} = \left\lceil \frac{p+q}{2} \right\rceil$$

key 25

mid

3	5	6	13	18	21	21	25	36	43	49	51	53	60	72	74	83	88	91	95
---	---	---	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----



If  $\text{key} \leq \text{mid}$ ,  
then key is in  
left half.

If  $\text{key} > \text{mid}$ ,  
then key is in  
right half.

# Boundary Conditions

$$\text{Select mid} = \left\lceil \frac{p+q}{2} \right\rceil$$

key 25

mid

If  $\text{key} \leq \text{mid}$ ,  
then key is in  
left half.

If  $\text{key} > \text{mid}$ ,  
then key is in  
right half.

# Boundary Conditions

- Another bug!

No progress  
toward goal:  
Loops Forever!

$$\text{Select } \text{mid} = \left\lceil \frac{p+q}{2} \right\rceil$$



If  $\text{key} \leq \text{mid}$ ,  
then key is in  
left half.

If  $\text{key} > \text{mid}$ ,  
then key is in  
right half.

# Boundary Conditions

```
mid =  $\left\lfloor \frac{p+q}{2} \right\rfloor$   
if key ≤ A[mid]  
    q = mid  
else  
    p = mid + 1  
end
```

OK

```
mid =  $\left\lceil \frac{p+q}{2} \right\rceil$   
if key < A[mid]  
    q = mid - 1  
else  
    p = mid  
end
```

OK

```
mid =  $\left\lceil \frac{p+q}{2} \right\rceil$   
if key ≤ A[mid]  
    q = mid  
else  
    p = mid + 1  
end
```

Not OK!!

# Getting it Right

- How many possible algorithms?
- How many **correct** algorithms?
- Probability of **guessing** correctly?

$mid = \left\lfloor \frac{p+q}{2} \right\rfloor$  ← or  $mid = \left\lceil \frac{p+q}{2} \right\rceil$  ?

if  $key \leq A[mid]$  ← or if  $key < A[mid]$  ?  
     $q = mid$

else  
     $p = mid + 1$  ← or  $q = mid - 1$   
end  
    else  
         $p = mid$   
    end

# Alternative Algorithm: Less Efficient but More Clear

BinarySearch( $A[1..n]$ ,  $key$ )

<precondition>:  $A[1..n]$  is sorted in non-decreasing order

<postcondition>: If  $key$  is in  $A[1..n]$ , algorithm returns its location

$p = 1, q = n$

while  $q \geq p$

<loop-invariant>: If  $key$  is in  $A[1..n]$ , then  $key$  is in  $A[p..q]$

$mid = \left\lfloor \frac{p+q}{2} \right\rfloor$

if  $key < A[mid]$

$q = mid - 1$

else if  $key > A[mid]$

$p = mid + 1$

else

return( $mid$ )

end

end

return("Key not in list")

Still  $\Theta(\log n)$ , but with slightly larger constant.

# Moral

- Use the loop invariant method to think about iterative algorithms.
- Be careful with your definitions.
- Be sure that the loop invariant is always maintained.
- Be sure progress is always made.
- Having checked the ‘typical’ cases, pay particular attention to boundary conditions and the end game.
- Sometimes it is worth paying a little in efficiency for clear, correct code.

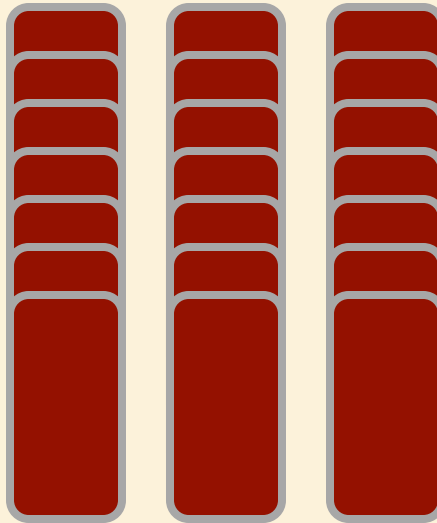
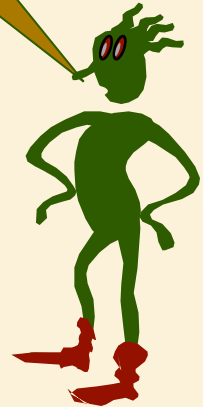
# Card Trick

➤ A volunteer, please.





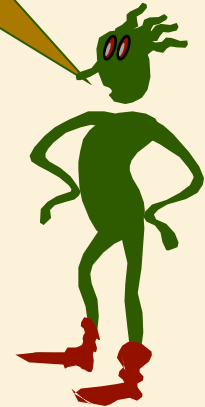
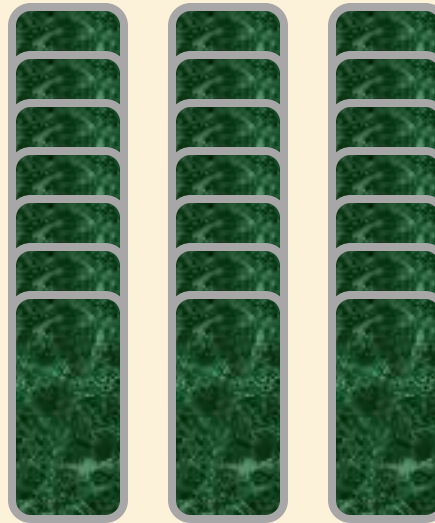
*Pick a Card*



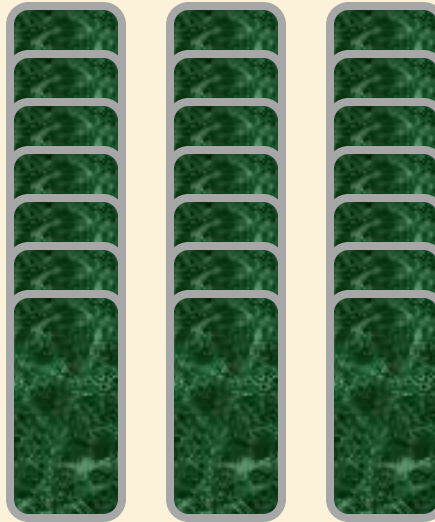
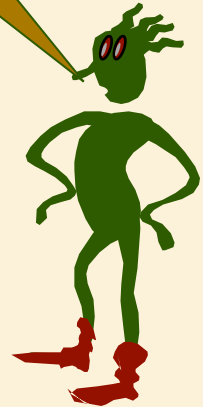
**Done**



***Loop Invariant:  
The selected card is one  
of these.***



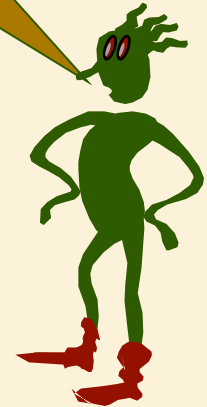
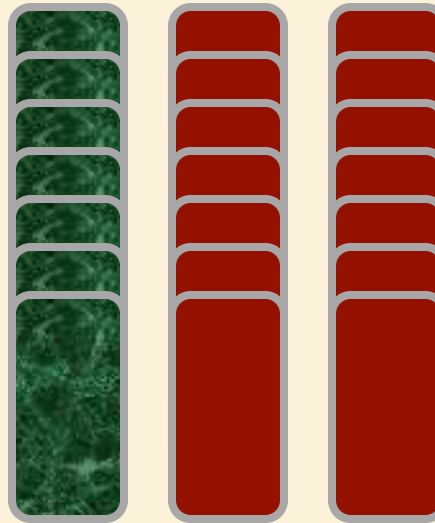
*Which  
column?*



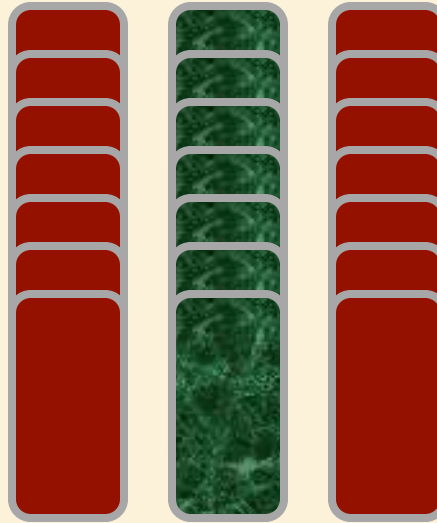
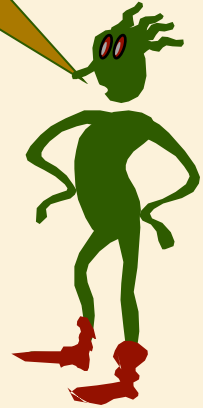
**left**



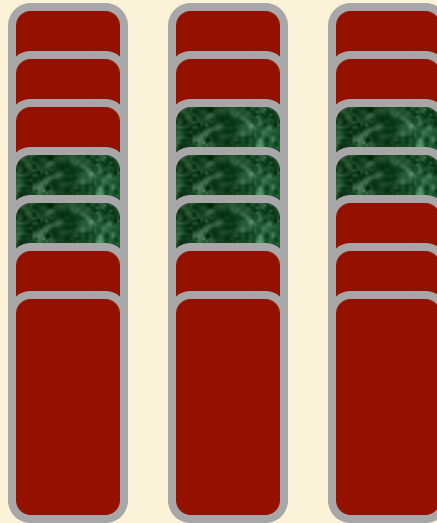
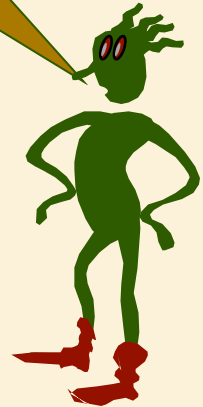
***Loop Invariant:  
The selected card is one  
of these.***



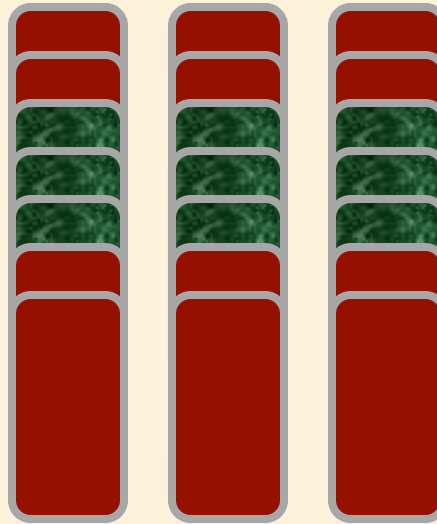
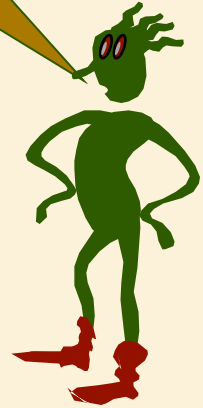
*Selected column is placed  
in the middle*



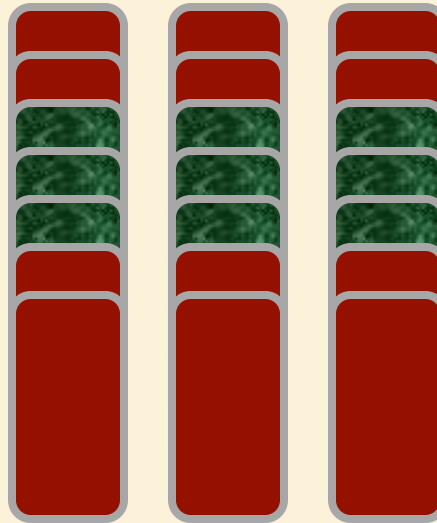
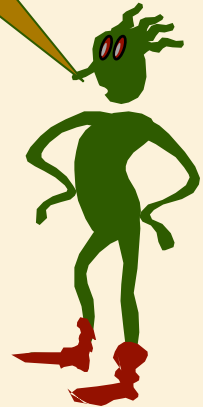
*I will rearrange the cards*



*Relax Loop Invariant:  
I will remember the same  
about each column.*



*Which  
column?*

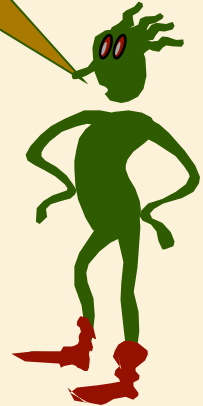
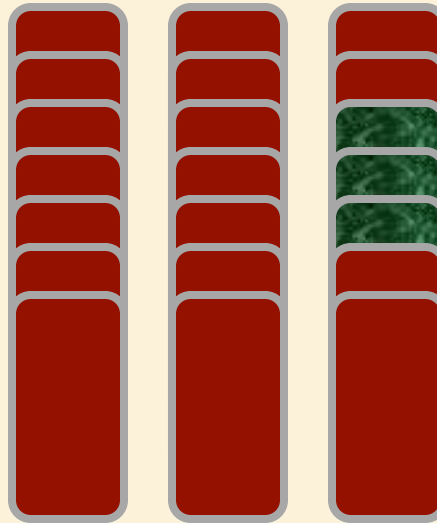


**right**

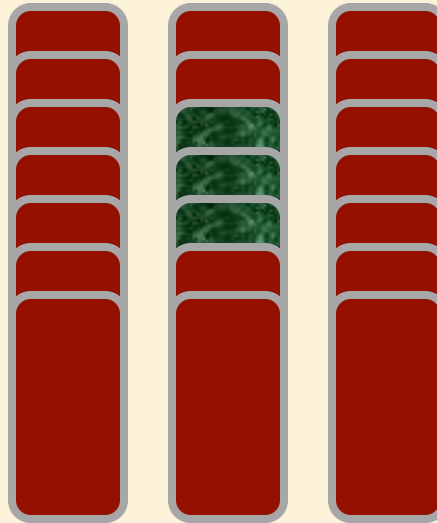
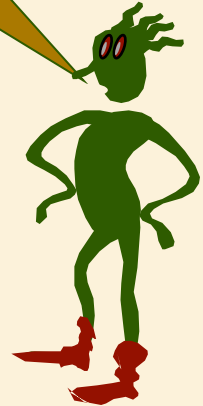




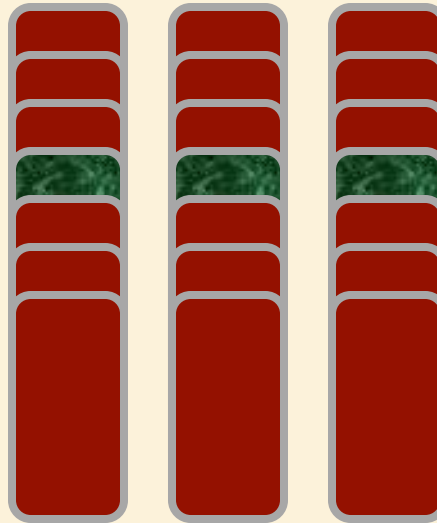
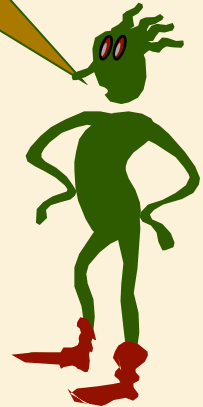
***Loop Invariant:  
The selected card is one  
of these.***



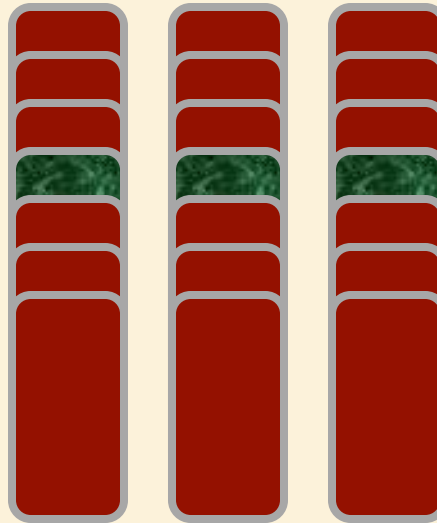
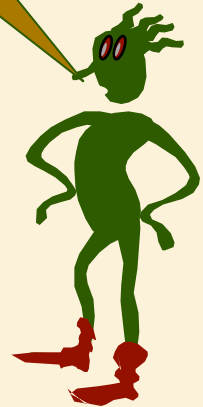
*Selected column is placed  
in the middle*



*I will rearrange the cards*



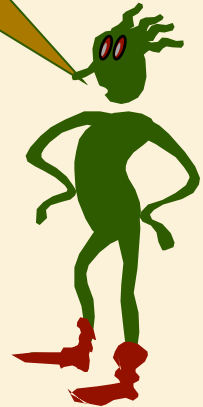
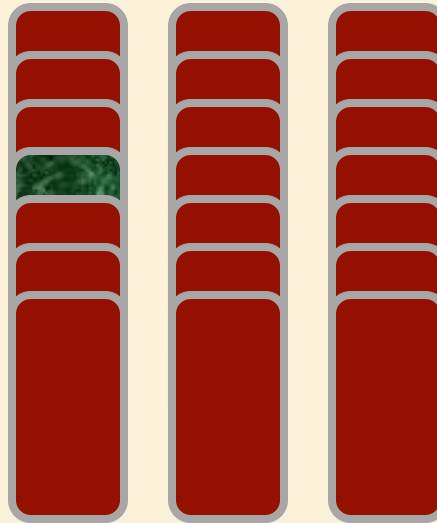
*Which  
column?*



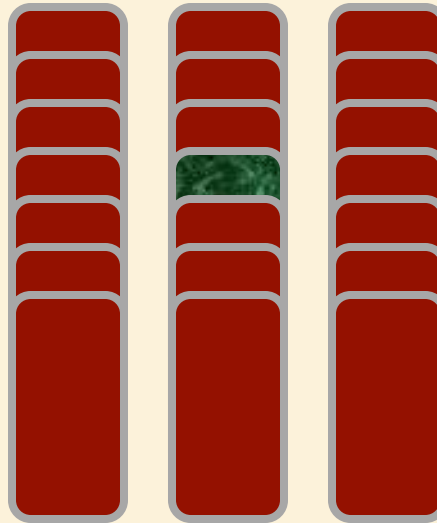
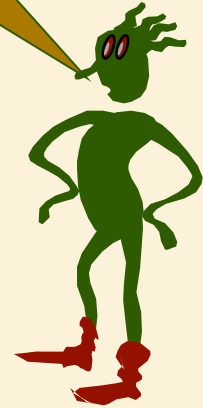
**left**



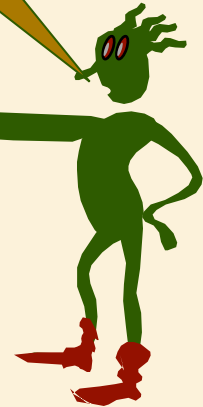
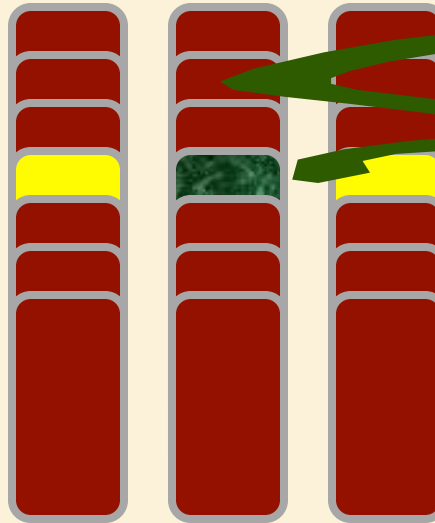
***Loop Invariant:  
The selected card is one  
of these.***



*Selected column is placed  
in the middle*



*Here is your  
card.*



**Wow!**



# Ternary Search

- **Loop Invariant:** selected card in central subset of cards

$$\text{Size of subset} = \lceil n / 3^{i-1} \rceil$$

where

$n$  = total number of cards

$i$  = iteration index

- How many iterations are required to guarantee success?



# Summary: Learning Outcomes

## ➤ Maps

- ❑ ADT
- ❑ What are they good for?
- ❑ Naïve implementation – running times

## ➤ Hashing

- ❑ Running time
- ❑ Types of hashing

## ➤ Dictionaries

- ❑ ADT
- ❑ What are they good for?

## ➤ Ordered Maps & Dictionaries

- ❑ Iterative Algorithms: Case Study of Binary Search
  - ✧ Loop Invariants
  - ✧ Boundary Conditions